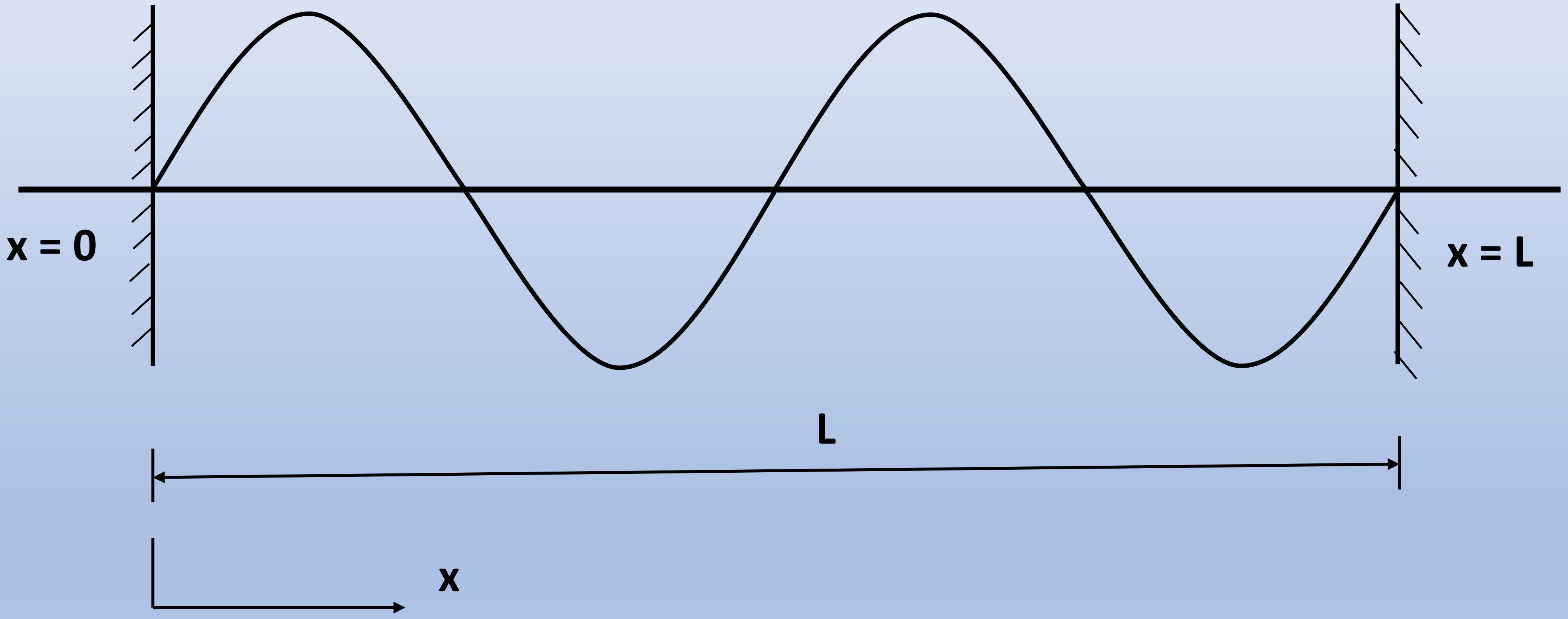
Solve 1D Wave Equation Using Finite Difference Method

Objectives

- Present a simple 1D Vibration problem, which is related to vibrations of an elastic string.
- The physical problem is described by 1D wave equation
- Solve the problem using Centered-Difference Finite Difference method.
- We will consider damped and undamped cases

1D Wave Equation Problem – Vibrations of an elastic string



1D Wave Equation

Assumptions and conditions :

- Thin flexible string with neglible weight
- The two ends of the string are clamped, so the displacements are zero at the ends
- We will consider two cases – one without damping and the other with damping
- Initial displacement profile is given
- Initial velocity is considered zero

1D Wave Equation

- $u_{tt} = c^2 * u_{xx}$ ………….(1);
- u(x, t); u – vertical displacement, c – wave propagation speed
- x – spatial coordinate, t – time; $0 <= x <= L$; $t > 0$;
- $c^2 = \left(\frac{T}{\rho}\right)$;
- T – force of tension exerted on the string ; ρ - mass density (mass per unit length of the string)
- Boundary Conditions:
- u(0,t) = 0; u(L,t) = 0;
- Initial Conditions:
- u(x,0) = f(x) ; u_t (x,0) = g(x)

1D Wave Equation – Simplified Case

- Simplified Case : $f(x) \neq 0$; $g(x) = 0$;
- $u_{tt} = c^2 * u_{xx}$ ....(1);
- Boundary Conditions:
- $u(0,t) = 0$; $u(L,t) = 0$;
- Initial Conditions:
- $u(x,0) = f(x) = c1 * \sin(c2 * \pi * x)$; $u_t(x,0) = 0$
- Example: If c1 = 1, c2 = 2, c = 0.5,
- $f(x) = \sin(2 * \pi * x)$

1D Wave Equation (Undamped) – Solution By Centered-Difference Method

- $u_{tt} = c^2 * u_{xx}$ ....(1);
- $\left(\frac{u_i^{n+1} - 2*u_i^n + u_i^{n-1}}{\Delta t^2}\right) = c^2 * \left(\frac{u_{i-1}^n - 2*u_i^n + u_{i+1}^n}{\Delta x^2}\right)$
- $\left(u_i^{n+1} - 2 * u_i^n + u_i^{n-1}\right) = \left(\frac{c^2 * \Delta t^2}{\Delta x^2}\right) * \left(u_{i-1}^n - 2 * u_i^n + u_{i+1}^n\right)$
- Let nu = $\left(\frac{c * \Delta t}{\Delta x}\right)$; where nu = courant/convection number;
- $\left(u_i^{n+1} - 2 * u_i^n + u_i^{n-1}\right) = nu^2 * \left(u_{i-1}^n - 2 * u_i^n + u_{i+1}^n\right)$
- $u_i^{n+1} = 2 * u_i^n - u_i^{n-1} + nu^2 * \left(u_{i-1}^n - 2 * u_i^n + u_{i+1}^n\right)$

1D Wave Equation (undamped) – Solution By Centered-Difference Method

- Initial Conditions:
- u_t (x,0) = 0
- $\left(\frac{u_i^{n+1} - u_i^n}{\Delta t}\right) = 0$
- $u_i^{n+1} = u_i^n$
- For n = 1 (1st time step)
- $u_i^2 = u_i^1$

1D Wave Equation (Damped) – Solution By Centered-Difference Method

- $u_{tt} + \Upsilon * u_t = c^2 * u_{xx}$ ....(1); Υ – damping factor
- $\left(\frac{u_i^{n+1} - 2*u_i^n + u_i^{n-1}}{\Delta t^2}\right) + \Upsilon * \left(\frac{u_i^{n+1} - u_i^{n-1}}{2 * \Delta t}\right) = c^2 * \left(\frac{u_{i-1}^n - 2*u_i^n + u_{i+1}^n}{\Delta x^2}\right)$ $(\times \Delta t^2)$
- $(u_i^{n+1} - 2 * u_i^n + u_i^{n-1}) + \left(\frac{\Upsilon * \Delta t}{2}\right) * (u_i^{n+1} - u_i^{n-1}) = \left(\frac{c^2 * \Delta t^2}{\Delta x^2}\right) * (u_{i-1}^n - 2 * u_i^n + u_{i+1}^n)$
- Let nu = $\left(\frac{c * \Delta t}{\Delta x}\right)$; where nu = courant/convection number;
- $(u_i^{n+1} - 2 * u_i^n + u_i^{n-1}) + \left(\frac{\Upsilon * \Delta t}{2}\right) * (u_i^{n+1} - u_i^{n-1}) = nu^2 * (u_{i-1}^n - 2 * u_i^n + u_{i+1}^n)$
- $u_i^{n+1} = \left(\frac{1}{1 + 0.5 * \Upsilon * \Delta t}\right)\left(2 * u_i^n - u_i^{n-1} + \left(\frac{\Upsilon * \Delta t}{2}\right) * u_i^{n-1} + nu^2 * (u_{i-1}^n - 2 * u_i^n + u_{i+1}^n)\right)$

1D Wave Equation – Solution By Centered-Difference Method

- Initial Conditions:
- $u_t\ (x,0) = 0$
- $\left(\frac{u_i^{n+1} - u_i^n}{\Delta t}\right) = 0$
- $u_i^{n+1} = u_i^n$
- For n = 1 (1st time step)
- $u_i^2 = u_i^1$

1D Wave Equation – Solution By Centered-Difference Method

- Centered–Difference method (space and time) is an explicit method and is conditionally stable.
- Stability criteria : $\mathrm{nu} = (\frac{c * \Delta t}{\Delta x}) <= 1$
- The error is $O(\Delta t^2) + O(\Delta x^2)$

Summary

In this video,

- We presented a 1D wave equation that describes the vibration of an elastic string
- We considered 2 cases – one without damping and one with damping.
- We solved the problem using Centered-Difference method.
- In future videos, we can explore more challenging problems.